Physics 200B **Mechanics II** **Winter 2013**

Problem Set III: Due February 28, 2013

1.) Use the product of the exterior product (wedge product) to prove total phase volume $d\mathrm{V}=d\underline{p}_1\, d\underline{q}_1\, d\underline{p}_2\, d\underline{q}_2 \ldots d\underline{p}_n\, d\underline{q}_n$ is conserved for a Hamiltonian system. Note: Start your solution by giving a more precise definition of $d\mathrm{V}$.

2.) Determine the stable equilibrium positions for a simple pendulum which oscillates:

a.) horizontally, with $x = x_0 \cos\ \omega t$

b.) in a circle, with $x = r_0 \cos\ \omega t$, $y = r_0 \sin\ \omega t$.

Take $\omega >> \sqrt{g/\ell}$ and consider the full range of parameters.

3.) Now again consider a simple pendulum with support oscillating at $y = y_0 \cos\ \omega t$. If the pendulum has length ℓ (so $\omega_0 = \sqrt{g/\ell}$) and $\omega = 2\omega_0 + \in$, determine the conditions for, and growth rate of, parametric instability.

4.) Consider a uniform magnetic field $B(t)\hat{z}$ whose strength is time dependent. Show that Faraday's law $\partial \underline{\mathrm{B}}/\partial t = -c\nabla \times \underline{\mathrm{E}}$, can be satisfied by an associated electric field

$$\underline{\mathrm{E}} = -\frac{\dot{B}}{2c}\left(x\hat{y} - y\hat{x}\right)$$

where the dot indicates a time derivative. Let any additional electrostatic field be zero.

Show that the equation of motion of an ion can be written in the form

$$\ddot{z} = -i\Omega\dot{z} - i\frac{\dot{\Omega}}{2}z$$

where $\Omega(t) = eB(t)/Mc$ and $z = x + iy$. Set

$$z(t) = w(t)\exp - i\int\left[\Omega(t)/2\right]dt \ .$$

and show that w satisfies the harmonic oscillator equation

$$\ddot{w}+\frac{\Omega^2(t)}{4}w=0 .$$

Let B be linear in time and show that one solution of this equation can be expressed in terms of ordinary Bessel function J. Take the limits $t\to\pm\infty$ and show that the magnetic moment of the ion is asymptoticly constant and the same in both limits, even though B has passed through zero and violated the usual condition for the conservation of the magnetic moment. If B is a constant, show that the motion is a circle in the $x-y$ plane.

5.) Compute the threshold for parametric instability in the presence of linear frictional damping, as well as mismatch. For what range of mismatch $\in$ will instability occur?

6.) Let $(q,p,t)=H_0(q,p)+V(q)d^2A/dt^2$ where $A(t)$ is periodic, with period $\tau<<T$. Here T is the period of the motion governed by H_0.

a.) Derive the mean field (i.e. short time averaged) equations for this system.

b.) Show that these mean field equations may be obtained from the effective Hamiltonian

$$K(p,q)=H_0(p,q)+\frac{1}{4m}\left\langle\left(\frac{dA}{dt}\right)^2\right\rangle\left(\frac{\partial V(q)}{\partial q}\right)^2 .$$

Here $<>$ means a short time average. You may assume $H_0=p^2/2m+V_0(q)$.

7.) What is the relation between the volume and pressure of a gas of point particles moving within a cubic box of side L? Assume the walls are hard, so particles reflect elastically, and that two opposite sides move together or apart *slowly*. Comment on the relation between your result and those of thermodynamics.

8.) Consider the asymmetric top, with moments of inertia $I_1 < I_2 < I_3$. Here 1, 2, 3 refer to the principal axes in a frame for which the inertia tensor is diagonal. Using the Euler equations:

a.) Derive the equations of motion for $\Omega_1(t)$, $\Omega_2(t)$, $\Omega_3(t)$, the angular frequencies associated with axes 1, 2, 3.

b.) Show that if $\Omega_2 \cong \Omega_0$ while Ω_1, Ω_2 start from an infinitesimal perturbation, instability results. Show that $\Omega_1 \cong \Omega_0$ or $\Omega_3 \cong \Omega_0$ is stable.

c.) What are the two conserved quantities which constrain the evolution in b.)?